Mechanism of the onset of axial segregation in a rotating cylindrical drum filled with binary granular mixtures

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A phenomenon of axial band formation in a long rotating cylindrical drum filled with a binary granular mixture is described as a two-stage process. At the first stage, due to different mobility of the large and small particles, a central core occupied by a small-size fraction is formed along the axis of the drum. The axial segregation, i.e., transverse band formation, is found to be associated with the instability of the radially segregated state caused by an asymmetry of the free surface of the granular material. This instability is caused by the difference in the repose angles of two granular fractions and deviation of the filling level of the drum from 0.5. [S1063-651X(99)04308-1]

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I. INTRODUCTION

Different aspects of granular material flow in partially filled and slowly rotating cylindrical drum have been studied intensively (see e.g., [1–11] and references therein) in view of numerous technological applications. An interesting phenomenon observed in a rotating drum is axial segregation of a binary granular mixture. When a bidisperse granular material partially fills a long horizontal rotating cylindrical drum, the components can segregate into alternating bands normal to the axis of the drum [1,3–7]. On the basis of their experiments with binary mixtures Donald and Roseman [1] and Hill and Kakalios [4] concluded that the axial segregation depends on the differences in the angles of repose of small (α_S) and large (α_L) particles. Segregation of the components occurred when $\alpha_S > \alpha_L$. The mechanism of transverse band formation was suggested to be associated with the axial gradients of the free surface height and the higher mobility of the large grains [3].

The first rigorous model of the axial segregation was proposed by Zik *et al.* [5]. Averaging the particle concentration in the transverse plane of the cylindrical drum, they derived an equation of mass conservation. It was assumed that since the different fractions have different angles of repose, the profile of the free surface depends on the composition of the granular mixture. Therefore, the variation of an averaged height of the free surface is proportional to a concentration, and the particle flux is proportional to the axial gradient of concentration. When particles with a smaller angle of repose have a large mobility down the slope, the equation of mass conservation reduces to a diffusion equation with a negative coefficient of diffusion. Thus axial segregation is caused by an instability similar to the phenomenon of spinodal decomposition. Note that variation of the free surface profile results in axial mass flow. In order to prevent granular material accumulation, the authors of [5] postulated a phenomenological counterflux which was assumed to be size nonpreferential. However, when an axial flow occurs due to the variation of the free surface profile caused by the concentrations gradients and granular material accumulates at the vicinity of some section of the drum, the counterflow is induced by the free surface elevation. Since the latter mechanism is size preferential, the segregation is eliminated.

Note also that the model of Zik *et al.* [5] neglects the granular segregation in the transverse plane of the drum which was observed in the majority of the experiments [1,6]. The authors of [5] assumed that a concentration depends only on the axial coordinate. It is still unclear why different physical properties of the granular components affect the mobility of the particles along the drum, but do not affect the mobility in the transverse direction, although the flow rate in the transverse plane is much larger.

Recently, the model of axial segregation proposed in [5] was substantially improved and developed in [10,11]. The authors considered a rapidly rotating long drum of a moderate radius where granular diffusion prevents the radial segregation. The dynamics of the surface granular flow was coupled with a bulk flow that prevents the granular material from accumulating anywhere. The dynamic angle of repose and composition of the granular material averaged over the cross section of the drum were the only variables of two one-dimensional differential equations which describe the initial stage of phase separation as well as the long-time nonlinear pattern evolution.

In the present study the onset of axial band formation is considered as a two-stage process. We assume, according to experimental observations [1,6], that when a characteristic length of the granular diffusion is smaller than the radius of the drum, axial segregation is preceded by radial segregation. The central core in the drum which is occupied by the smaller or heavier particles can be stable or unstable depending on the angles of repose of the granular materials and the filling level of the drum. Thus we associate the mechanism of the onset of transverse band formation with the instability of the radially segregated state. The instability of the radially segregated state implies that a uniform distribution of two fractions of the granular material along the axis of the drum

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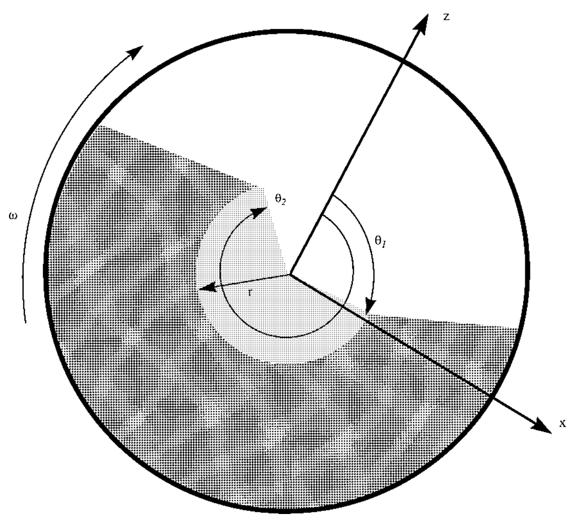

FIG. 1. Schematic view of the flow of granular mixture in a transverse plane of a rotating cylindrical drum and the coordinates system.

is not possible. Thus some modulation of the concentration along the cylinder axis will develop whereby the concentration becomes a function of an axial coordinate.

II. ONSET OF AXIAL SEGREGATION IN A ROTATING CYLINDRICAL DRUM

Consider a long cylindrical drum with radius 1 rotating with a constant angular velocity ω (Fig. 1). The drum is filled with a mixture of two different granular materials, e.g., with large and small grains. The y axis is directed along the cylinder; the x axis is inclined at an angle α with respect to the horizontal. The angle α is interpreted as the mean angle of the inclination of the free surface. Denote the profile of the free surface as $z=h(t,x,y)$. Assume also that the granular flow occurs in a thin surface layer.

Since small grains are trapped by the surface roughness more easily than the larger ones, after a few rotations the small particles accumulate near the axis of the cylinder. Magnetic resonance measurements demonstrate that even when the granular material seems mixed at the surface, there is a radially segregated core of small particles inside the bulk [6]. Here we assume that this radial segregation is complete, i.e., that there is a central core with a radius r which contains only small particles, while the large particles occupy the outer region. In order to examine the stability of the central core with respect to the axial perturbations, we adopt the following model of granular flow in the transverse plane. The equation of material balance reads

$$\frac{dh}{dt}+\frac{\partial q_0}{\partial x}\equiv\frac{\partial h}{\partial t}+V_x\frac{\partial h}{\partial x}-V_z+\frac{\partial q_0}{\partial x}$$
$$\equiv\omega\left(h\,\frac{\partial h}{\partial x}+x\right)+\frac{\partial q_0}{\partial x}=0,$$

where q_0 is a volumetric flow rate of granular material in the x direction. Integration of the above equation with the boundary condition $q_0(x^2+h^2=1)=0$ yields the following formula which was previously obtained in [2]: $q_0(x)=q_0^S(x)+q_0^L(x)=\omega/2[1-x^2-h^2(x)]$, where $q_0^S(x)$ and $q_0^L(x)$ are the volumetric flow rates of the small- and large-size particle fractions, respectively,

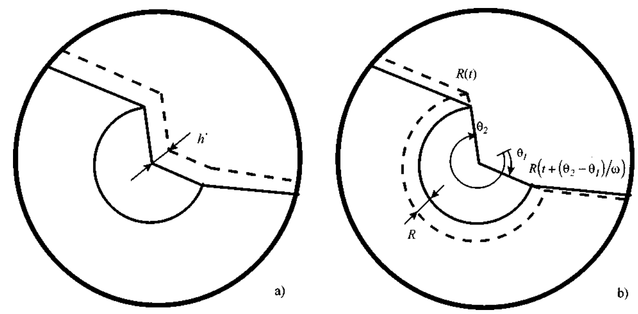

FIG. 2. Free surface elevation.

$$q_0^S(x)=\max\left(0,\frac{\omega}{2}[r^2-x^2-h^2(x)]\right), \tag{1}$$

$$q_0^L(x)=q_0(x)-q_0^S(x). \tag{2}$$

We assume that the stationary bulk is stable until it is inclined at a maximum angle of stability, α_m, while the free surface flow vanishes when the free surface is inclined at a minimum angle of repose, α_r ($\alpha_m>\alpha_r$) [12]. When the small grain fraction is more coarse, the slope of a free surface at the center of the drum is larger than near its walls. Thus we assume that there are four regions where the free surface is flat and where is inclined at different angles with respect to the horizontal: α_{mL}, α_{mS}, α_{rS}, and α_{rL} (see Fig. 1 from left to right), where $\alpha_{mS}>\alpha_{mL}$, $\alpha_{mS}>\alpha_{rS}$, and $\alpha_{rS}>\alpha_{rL}$. This free surface profile can be viewed as a crude approximation to the S-shaped free surface observed in the experiments [2]. Certainly, the real free surface is smooth due to material composition continuity. However, the characteristic size of the transitional region where the discontinuity in a material composition is smoothed is of the order of a few grain size [13,14] and it can be neglected in the present model.

We analyze the linear stability of the system assuming that (see Fig. 2): $r=r_0+R(t,y)$, $h=h_0(t,x)+\tilde{h}(t,y)+H[(R)]$, where $R(t,y)$ is a perturbation of the radius of the central core, $\tilde{h}(t,y)$ is an x-independent perturbation of the free surface height [Fig. 2(a)], and $H[(R)]$ is a functional which accounts for the perturbation of the free surface due to the change of the radius [Fig. 2(b)]. Note that while the free surface at the upper part of the drum depends on a value of a radius perturbation $R(t,y)$, the free surface in the lower part of the drum depends on $R(t+(\theta_2-\theta)_1/\omega,y)$. Then expanding the radius into power series with respect to time, we find that $H[(R)]\approx H(R,\partial R/\partial t)$ and $R(t-\Delta\theta/\omega,y)\approx R(t,y)+(\pi/\Delta\theta)\partial R/\partial t$. This approximation is valid when a characteristic time of the excitation of the instability is large compared with the period of the drum rotation, i.e., $1/\lambda\gg 2\pi/\omega$, where λ is the growth rate of perturbations.

Denote the volumes of the small- and large-size particle fractions per unit length of the drum V_S and V_L, respectively. Then the change of a volume of small particles fraction in the control volume between two cross section of the drum $(y,y+dy)$ reads

$$\frac{\partial V_S}{\partial t}=L\delta\frac{\partial \tilde{h}}{\partial t}+\beta\frac{\partial R}{\partial t}. \tag{3}$$

The first term in Eq. (3) accounts for the change of a volume of small particle fraction due to perturbation of the free surface, and the second term describes the variation of a volume small particle fraction as a result of the radius perturbation (see Fig. 2). Assuming that the free surface is nearly flat (α_{mS}, α_{mL}, α_{rS}, $\alpha_{rL}\approx\alpha$) and that the drum is nearly half filled, we obtain

$$\frac{\partial V_S}{\partial t}=L\delta\frac{\partial \tilde{h}}{\partial t}+\beta\frac{\partial R}{\partial t}\approx\frac{\partial}{\partial t}\left(2r\tilde{h}+\frac{1}{2}\int_{\theta_1}^{\theta_2}r(\theta,y)^2d\theta\right)$$

$$\approx 2r\frac{\partial \tilde{h}}{\partial t}+\omega r\left[R(t,y)-R\left(t-\frac{\pi}{\omega},y\right)\right]$$

$$\approx 2r_0\frac{\partial \tilde{h}}{\partial t}+\pi r_0\frac{\partial R}{\partial t}.$$

Thus the parameters in the above equation can be estimated as $L\delta\approx 2r_0$, where $L\approx 2$, $\delta\approx r_0$, and $\beta\approx\pi r_0$. Similarly, we find that

$$\frac{\partial V_L}{\partial t}=L(1-\delta)\frac{\partial \tilde{h}}{\partial t}-\beta\frac{\partial R}{\partial t}. \tag{4}$$

We assume that the volumetric flow rate of each component in the mixture in the y direction is proportional to the free surface inclination, and the total flow rate in the axial direction reads

$$Q=-\cos(\alpha)\int q\frac{\partial h}{\partial y}dx$$

$$=-\cos(\alpha)\int(q_0+q')\frac{\partial h}{\partial y}dx$$

$$\approx-\cos(\alpha)\int q_0\frac{\partial h}{\partial y}dx,$$

where $q=q_0(x)+q'(t,x,y)$ is a proportionality coefficient and in the linearized equations the term q' can be omitted. Integration of Eq. (1) yields the total flow rate of the small-size particle fraction:

$$Q_S=-\cos(\alpha)\int q_0^S\frac{\partial h}{\partial y}dx=-\gamma D\frac{\partial \tilde{h}}{\partial y}, \tag{5}$$

where γ and D can be approximated as $\gamma\approx r_0^3$, and $D\approx(\omega/3)\cos(\alpha)$. Note that the axial flux of the small particles does not depend on R because the variation of the central core radius affects the height of the free surface only in the outer region [Fig. 2(b)]. Integration of Eq. (2) yields the total flow rate of the large particles fraction:

$$Q_L=-\cos(\alpha)\int q_0^L\frac{\partial h}{\partial y}dx$$

$$=-(1-\gamma)D\frac{\partial \tilde{h}}{\partial y}-W\frac{\partial R}{\partial y}+A\frac{\partial^2 R}{\partial t\partial y}, \tag{6}$$

where the two last term describe the free surface elevation due to a change of the radius of the central core and the coefficients W and A are given by the expressions

$$W=\cos(\alpha)\int q_0^L\frac{\partial H}{\partial R}dx,\quad A=-\cos(\alpha)\int q_0^L\frac{\partial H}{\partial(\partial R/\partial t)}dx.$$

The expressions for the coefficients W and A are rather involved and will be specified below. Note that all the coefficients L, δ, γ, β, D, and A are positive. When the material flux q_0 and a free surface inclination $\partial h/\partial x$ are symmetric with respect to the center of the drum, $W\equiv 0$. Thus W can be positive or negative depending on the symmetry of the free surface profile. Combining Eqs. (3)–(6), we obtain the following equation of continuity for small and large grains fractions, respectively:

$$L\delta\frac{\partial \tilde{h}}{\partial t}+\beta\frac{\partial R}{\partial t}=\gamma D\frac{\partial^2 \tilde{h}}{\partial y^2}, \tag{7}$$

$$L(1-\delta)\frac{\partial \tilde{h}}{\partial t}-\beta\frac{\partial R}{\partial t}=(1-\gamma)D\frac{\partial^2 \tilde{h}}{\partial y^2}+W\frac{\partial^2 R}{\partial y^2}-A\frac{\partial^3 R}{\partial t\partial y^2}.$$

The sum of the above equations yields the equation of material balance in the y direction:

$$L\frac{\partial \tilde{h}}{\partial t}=D\frac{\partial^2 \tilde{h}}{\partial y^2}+W\frac{\partial^2 R}{\partial y^2}-A\frac{\partial^3 R}{\partial t\partial y^2}. \tag{8}$$

We seek a solution to Eqs. (7) and (8) in the form

$$\tilde{h}=\tilde{h}_0\exp(\lambda t+iky),$$

$$R=R_0\exp(\lambda t+iky), \tag{9}$$

where k is the wave number. Substituting Eqs. (9) into Eqs. (7) and (8) yields

$$\lambda L\delta\tilde{h}_0+\lambda\beta R_0=-k^2\gamma Dh_0,$$

$$\lambda L\tilde{h}_0+k^2D\tilde{h}_0=-k^2WR_0+\lambda k^2AR_0.$$

After some algebra we obtain the dispersion equation

$$L(\beta+k^2\delta A)\lambda^2+k^2(D\beta+k^2A\gamma D-L\delta W)\lambda-k^4\gamma DW=0, \tag{10}$$

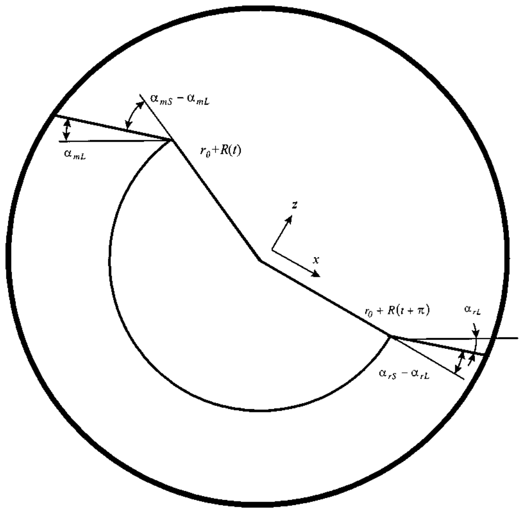

FIG. 3. Profile of the free surface in a half-filled drum.

which can be easily solved with respect to the growth rate λ. Since all the coefficients except for *W* in the dispersion equation (10) are positive, the only parameter which determines the stability of the radial segregated state is *W*. When $W=0$ (a perfectly symmetric free surface inclination profile), one of the roots of Eq. (10) is zero, while the other is negative. For $W>0$ there is a positive root of Eq. (10), and the initial state is unstable. When *W* changes its sign, the radial segregation becomes stable and the alternating transverse bands are not formed. The mechanism of the instability of the initially radially segregated state is as follows. When the radius of the central core at a transverse section of the cylinder increases, i.e., when the number of small particles grows, elevation of the free surface at the upper part of the drum causes outward flux of the large-size particle fraction. Thus the small-size particle fraction squeezes out the large-size particle fraction. Contrarily, at the lower part of the drum the height of the free surface decreases, and large particles flow into the control volume and the instability is suppressed. Thus the coefficient *W*, which depends on the asymmetry of the initial state, characterizes the difference between these two competing processes. Note that Eq. (10) implies that the instability growth rate λ is positive for $k\to\infty$. The reason for this behavior is the assumption of the complete segregation of two fractions. Incomplete segregation at the interface between the fractions will cause a damping of small corrugation perturbations of the interface due to mutual diffusion of the components.

We discuss now a possible cause for the instability. Consider a half-filled drum (Fig. 3). Assume that the free surface inclination is nearly constant: $|\alpha-\tan^{-1}(\partial h/\partial x)|\leq\varepsilon\ll 1$. In this case the particle flux with first-order accuracy of the small parameter ε is given as $q_0=(\omega/2)\,(1-x^2)$. The free surface elevation due to change of the radius of the central core in the lower and upper parts of the drum is given by $-R(t)\sin(\alpha_{mS}-\alpha_{mL})\cos(\alpha_{mL})$ and $R(t+\pi/\omega)\sin(\alpha_{mS}-\alpha_{mL})\cos(\alpha_{mL})$, respectively, while in the central part of the drum it is zero. Thus the equation for the volumetric flow rate of a large-size grain fraction induced by the radius perturbation reads

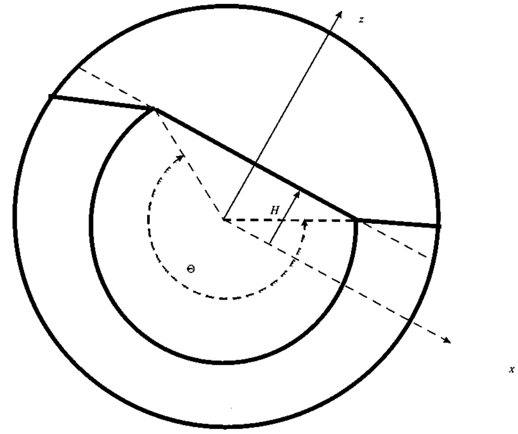

FIG. 4. Profile of the free surface in a more than half-filled drum.

$$W\frac{\partial R}{\partial y}-A\frac{\partial^2 R}{\partial t\partial y}=\frac{\partial}{\partial y}\Bigg\{R(t)\frac{\omega}{2}\int_{-1}^{-r_0}(1-x^2)dx\sin \times(\alpha_{mS}-\alpha_{mL})\cos(\alpha_{mL})-R\left(t+\frac{\pi}{\omega}\right)\frac{\omega}{2} \times\int_{r_0}^{1}(1-x^2)dx\sin(\alpha_{rS}-\alpha_{rL}) \times\cos(\alpha_{rL})\Bigg\}.$$

Approximating $R(t+\pi/\omega)\approx R(t)+(\pi/\omega)\partial R(t)/\partial t$, we obtain

$$W=\frac{\omega}{6}(2-3r_0+r_0^3)[\cos(\alpha_{mL})\sin(\alpha_{mS}-\alpha_{mL}) -\cos(\alpha_{rL})\sin(\alpha_{rS}-\alpha_{rL})],$$

$$A=\frac{\pi}{6}(2-3r_0+r_0^3)\cos(\alpha_{rS})\sin(\alpha_{rS}-\alpha_{rL}). \qquad (11)$$

Consequently, *A* is always positive, while *W* can change its sign depending on the combination of the angles of repose; i.e., when $\cos(\alpha_{mL})\sin(\alpha_{mS}-\alpha_{mL})-\cos(\alpha_{rL})\sin(\alpha_{rS}-\alpha_{rL})>0$, the radial segregation is unstable. Indeed, the free surface in the upper part of the drum where the motion initiates has a sharp profile, while in the lower part it is smoothed due to the inertia of the falling particles [2], and one can expect that the difference $\alpha_{mS}-\alpha_{mL}$ is always larger than $\alpha_{rS}-\alpha_{rL}$.

Consider now an idealized case when each of the granular materials can be characterized by one angle of repose, i.e., $\alpha_{mS}=\alpha_{rS}=\alpha_S$ and $\alpha_{mL}=\alpha_{rL}=\alpha$, but the drum more (less) than half-filled (Fig. 4). We also assume that $\alpha_S-\alpha=\varepsilon\ll 1$. The height of the free surface at the center of the cylinder is *H*, the length of the free surface is $2L_0$, and the length of the free surface of the small size fraction core is $2d$.

The coefficients A and W can be obtained from the following expressions:

$$W\frac{\partial R}{\partial y}-A\frac{\partial^2 R}{\partial t\partial y}$$
$$=\frac{\partial}{\partial y}\left\{R(t)\int_{-L_0-\varepsilon(L_0-d)H/L_0}^{-d}\{1-x^2-[H+\varepsilon(x+d)]^2\}\right.$$
$$\times dx-R\left(t+\frac{\Theta}{\omega}\right)\int_{d}^{L_0-\varepsilon(L_0-d)H/L_0}$$
$$\left.\times\{1-x^2-[H+\varepsilon(x-d)]^2\}dx\right\}$$
$$\times\frac{\omega}{2}\cos(\alpha)\sin(\alpha_S-\alpha_L).$$

Keeping only the lower-order terms in ε yields

$$W=\omega H(\alpha_S-\alpha_L)^2\cos(\alpha)(L_0-d)^2,$$
$$A=\frac{\Theta}{6}(2-3d+d^3)\cos(\alpha)\sin(\alpha_S-\alpha_L). \tag{12}$$

For the more than half-filled drum, the height of the free surface H is positive (see Fig. 4), and the radial segregation is unstable. For the less than half filled drum, the instability is not excited. Notably, the expression for W given by Eqs. (12) is of second order in $(\alpha_S-\alpha_L)$, while in Eqs. (11), W is of first order with respect to $\cos(\alpha_{mS})\sin(\alpha_{mS}-\alpha_{mL})-\cos(\alpha_{rS})\sin(\alpha_{rS}-\alpha_{rL})$. Therefore, in the case when the two causes for the asymmetry which leads to the instability, namely, different angles of repose of both fractions and a filling level, occur simultaneously, i.e., in a more than half-filled drum, the axial segregation is excited more easily. The condition for the excitation of the axial segregation in a less than half-filled drum is more restrictive since the only mechanism of segregation is the difference in the repose angles of the fractions.

III. CONCLUSIONS

In summary, the phenomenon of axial band formation in a long rotating cylindrical drum filled with a binary granular mixture was described as a two-stage process. At the first stage, due to different mobilities of the large and small particles, a central core occupied by the small-size fraction is formed along the axis of the drum. The onset of axial segregation is found to be associated with the instability of the radially segregated state. The instability is caused by an asymmetry of the free surface which arises due to the differences of the repose angles of the components (e.g., large- and small-size fractions) and the filling level of the drum. In a more than a half-filled drum, the axial segregation occurs more easily in comparison with a less than a half-filled drum.

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